C211118 Reg. No :

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALICUT
Model Question Paper
PhD. Entrance Examination, 2023

Time: 2 Hours Max Marks :100

PART A

Answer **all questions**, each question carries **2 marks**
(Notations: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ represent respectively the sets of integers, rational numbers, real numbers and complex numbers; PO set = partially ordered set)

($25 \times 2 = 50$ marks)

1. According to Redman and Mory Research is a
   - (a) a careful investigation or inquiry specially through search for new facts in any branch of knowledge
   - (b) systematized effort to gain new knowledge
   - (c) search for knowledge
   - (d) scientific and systematic search for pertinent information on a specific topic

2. Suppose that the statement "P and Q" is not true. Then which one of the following is true?

   (a) P implies Q (b) Q implies P
   (c) not P or not Q (d) none of the above

3. The relation $R = \{(a, b) : a - b$ is divisible by $4\}$ is

   (a) reflexive but not transitive (b) reflexive, symmetric and transitive
   (c) reflexive but not symmetric (d) reflexive and symmetric but not transitive

4. Zorn's lemma assures that
   - (a) every partially ordered set has a maximal element
   - (b) every partially ordered set has a minimal element
   - (c) every partially ordered set is well ordered
   - (d) None of the above

5. Which of the sets: $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{C}$ are equivalent?

   (a) $\mathbb{R}$ and $\mathbb{Q}$ (b) $\mathbb{Q}$ and $\mathbb{C}$ (c) $\mathbb{Z}$ and $\mathbb{Q}$ (d) $\mathbb{Z}$ and $\mathbb{R}$

6. Which one of the following sets is countable?

   (a) The set of all functions from the set $\mathbb{Z}$ of all integers to itself
   (b) Any countable product of countable sets
   (c) The set of all sequences whose terms are either 0 or 1
   (d) None of the above

7. If the collection of all algebraic extensios of the field of rational numbers $\mathbb{Q}$ is partially ordered with the usual inclusion relation " $\subseteq$ ", then which one of the following is a maximal element of this PO set?

   (a) $\mathbb{R}$ (b) $\mathbb{C}$ (c) the algebraic closure of $\mathbb{Q}$ (d) None of these

8. Let V and W be vector spaces over $\mathbb{F}$ with ordered bases $\mathfrak{B} = \{\alpha_1, \ \alpha_2, \ \alpha_3\}$ and $\mathfrak{B}' = \{\beta_1, \ \beta_2, \ \beta_3\}$ and let $T : V \to W$ be the linear map with $T(\alpha_i) = \beta_i, \ i = 1, 2, 3$. Then $[T]_{\mathfrak{B},\mathfrak{B}'}$ is

   (a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (d) none of these

9. Let V and W be two vector spaces over $\mathbb{F}$ and $T : V \to W$ be linear. Then
   (a) T is onto $\Rightarrow$ T is one one (b) T is one one $\Rightarrow$ T is onto
   (c) T is nonsingular $\Rightarrow$ T is onto (d) none of the above

10. The maximum order of an element in the Symmetric group S_{13} is
    (a) 13 (b) 40 (c) 42 (d) 36

11. Let $\mathbb{Q}[x]$ be the polynomial ring in x with rational coefficients and $I =< x^4 - 2x^2 - 2 >$. Which one of the following is true?
    (a) I is a maximal ideal (b) I is a prime ideal but not a maximal ideal
    (c) I is not a prime ideal (d) $R[x]/I$ has zero divisors

12. Which one of the following subsets of $\mathbb{R}^2$ is connected?
    (a) $\mathbb{Q} \times \mathbb{Q}$ (b) $(\mathbb{R} \setminus \mathbb{Q}) \times (\mathbb{R} \setminus \mathbb{Q})$
    (c) $(\mathbb{R} \setminus \mathbb{Z}) \times (\mathbb{R} \setminus \mathbb{Z})$ (d) none of the above

13. Let $f : [0, 1] \to \mathbb{R}$. Consider the following statements: On $[0, 1]$, f is (i) continuous (ii) uniformly continuous (iii) differentiable (iv) Riemann integrable
    Which one of the following is not true?

    (a) $(i) \Rightarrow (ii) \Rightarrow (iv)$ (b) $(i) \Rightarrow (iv) \Rightarrow (iii)$
    (c) $(iii) \Rightarrow (ii) \Rightarrow (i)$ (d) $(iii) \Rightarrow (ii) \Rightarrow (iv)$

14. Let $f_n : [0, 1] \to \mathbb{R}$ be the map $f_n(x) = \frac{x}{n}, \ n = 1, 2, 3, ...$ then f_n converges to the function $f : [0, 1] \to \mathbb{R}$ where
    (a) $f(x) = 1$ (b) $f(x) = 0$ (c) $f(x) = x$ (d) f_n does not converge

15. Let τ_1 and τ_2 be two topologies on a set X. Then which one of the following is not true?

    (a) The identity function from (X, τ_1) to (X, τ_2) is continuous iff $\tau_2 \subseteq \tau_1$

    (b) The identity function from (X, τ_1) to (X, τ_2) is a homeomorphism iff $\tau_1 = \tau_2$

    (c) (X, τ_1) and (X, τ_2) are homeomorphic iff $\tau_1 = \tau_2$

    (d) None of the above

16. Which one of the following is true?
    (a) Every regular space is Hausdorff
    (b) Every regular space is normal
    (c) Every normal space is regular
    (d) None of the above

17. Which one of the following space has a non constant continuous real valued function on it?
    (a) Sierpinski space
    (b) Cofinite space on an infinite set
    (c) Cocountable space on an uncountable set
    (d) None of the above

18. The number of roots of the polynomial $z^7 - 5z^3 + 12$ lie between the circles $|z| = 1$ and $|z| = 2$ is
    (a) 3 (b) 7 (c) 2 (d) 0

19. Let $f(z) = \frac{1}{z^2+3z+3}$. Then the coefficients of $\frac{1}{z^3}$ in the Laurent series expansion of $f(z)$ for $|z| > 3$.
    (a) 0 (b) -3 (c) -4 (d) -1

20. Consider the nonhomogeneous equation

$$y'' + 6y' + 9y = (2t + t^4)e^{-3t}$$

    By the method of undetermined coefficients, there is a solution to the equation which is of the form $y = Q(t)e^{-3t}$ where $Q(t)$ is a polynomial. The degree of $Q(t)$ is
    (a) 6 (b) 2 (c) 3 (d) 1

21. The value of λ for which the integral equation

$$y(x) = \lambda \int_0^1 (6x - \xi) y(\xi) d\xi$$

    has a non trivial solution, are given by the roots of the equations:
    (a) $(3\lambda - 1)(2 + \lambda) - \lambda^2 = 0$
    (b) $(3\lambda - 1)(2 + \lambda) - 4\lambda^2 = 0$
    (c) $(3\lambda - 1)(2 + \lambda) + \lambda^2 = 0$
    (d) $(\lambda - 1)(2 + \lambda) - \lambda^2 = 0$

22. The solution of the PDE, $xu_x + yu_y = \alpha u$, and $u = \phi(x)$ on the initial curve $y = 1$ is
    (a) $u = \phi(x/y)y^{\alpha}$ (b) $u = \phi(y/x)y^{-\alpha}$ (c) $u = \phi(x/y)y^{-\alpha}$ (d) $u = \phi(y/x)y^{\alpha}$

23. Let f be a continuous linear transformation from a normed space X into a normed space Y. Which one of the following is not true?
    (a) f is bounded
    (b) f is uniformly continuous
    (c) f is Lipschitz continuous
    (d) the zerospace $Z(f)$ is always nondense

24. Which one of the following is not true?
    (a) l^2 is a separable Hilbert space
    (b) any separable infinite dimensional Hilbert space is isometric to l^2
    (c) there is a countable orthonormal basis for l^2
    (c) there is a countable Hamel basis for l^2

25. Let G be a graph on n vertices and m edges. Consider the following statements:
    (i) G is connected $\quad$ (ii) $m \geq n-1$ $\quad$ (iii) G has a spanning tree
    Which one of the following is not true?

    (a) $(i) \Rightarrow (ii)$ $\quad$ (b) $(ii) \Rightarrow (iii)$ $\quad$ (c) $(iii) \Rightarrow (ii)$ $\quad$ (d) $(i) \Rightarrow (iii)$

PART - B

Answer any **10 questions**. Each question carries **5 marks**

($10 \times 5 = 50$ marks)

26. What are the Major differences between Research methods and Research methodology?

27. Write a short note on the components of a research problem.

28. Let X and Y be non-empty sets and f be a function from X into Y. Then show that f is one-to-one if and only if there exists a function g from Y into X such that $g \circ f = I_X$.

29. What is the Principle of Mathematical Induction? Explain with an example that how this principle is used as a proof technique in mathematics.

30. Give three examples of operations that can be defined on a set. Illustrate with examples.

31. Let S be a subspace of a vector space V. Show that $\sim$ is an equivalence relation on V where $x \sim y$ if and only if $x - y \in S$. What are the equivalence classes?

32. Prove that every number of the form $2^{a-1}(2^a - 1)$ is perfect if $2^a - 1$ is prime. Prove that if n is even and perfect then $n = 2^{a-1}(2^a - 1)$ for some $a \geq 2$.

33. Find the integral surface of the differential equation $uu_x + u_y = 1$ with the initial condition $x = s, y = s, u = s/2, 0 \leq s \leq 1$.

34. Give an example of a graph having automorphism group isomorphic to the Klein 4-group.

35. Prove that if α and β are algebraic over the field F, then $\alpha + \beta$ is algebraic over F.

36. Prove or disprove: A continuous function $f : [0,1] \to \mathbb{R}$ attains its maximum and minimum.

37. Let f be an analytic function on a region G to $\mathbb{C}$ except for poles. Show that the poles of f cannot have a limit point in G.

38. Let D be the open unit disk. Does there exist an analytic function $f : D \to D$ with $f(\frac{1}{2}) = \frac{3}{4}$ and $f^{'}(\frac{1}{2}) = \frac{2}{3}$?

39. Let X and Y be normed spaces and $F : X \to Y$ be linear. Then prove that F is continuous if and only if every cauchy sequence (x_n) in X, the sequence $(F(x_n))$ is Cauchy in Y.

40. Find all matrices which commute with $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

41. Let (X, d) be a metric space and A be a subset of X. let f be the real valued function defined on X by $f(x) = inf\{d(x,a) : a \in A\},\ x \in X$. Then prove that f is continuous and that $f(x) = 0$ if and only if $x \in cl(A)$.

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PhD Entrance Syllabus
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALICUT
With effect from 2022 admission

Unit - I : Research Methodology

1 Research Methodology

Foundations of Research, Problem Identification and Formulation, Research Design, Qualitative and Quantitative Research, Measurement

Research Aptitude: Research meaning, ethics and characteristics, Type of Research, Methods of Research, and Thesis Writing: Its Characteristics and Format

Reasoning: Mathematical reasoning, Numerical reasoning, Arguments, deductive and inductive research, Logical and Venn diagram, Inferences, Analogies

Data Interpretation: Interpretation of data, Mapping analysis of data, Quantitative and qualitative research

Language Comprehension

2 Set Theory and Logic and Proof Techniques used in Mathematics

Basics of Set theory, Ordered pairs, Relation and functions, one-to-one correspondence, Composition and inversion for functions, Operation for collection of sets, equivalent sets, cardinal number, finite and infinite sets, Countable and uncountable sets with examples, Ordering relations,Order types and Well ordered sets and ordinal numbers

History and basics of Number theory, congruences, Arithmetic functions

Axiom of choice, Well ordering Theorem and Zorn's lemma, Principles of Mathematical Induction, Pigeonhole principle, Newtons Binomial Theorem, Archimedes determination of circular Area, Fibonacci and the Rabit problem

Elementary logic, Logical statement and quantifiers, Operations on propositions, Truth tables, equivalent statements and tautologies, conditional and bi-conditional statements

Topics in Discrete Mathematics: Graph theory, Combinatorics

Unit - II : Advanced Mathematics

3 Algebra

Groups, Subgroups, abelian groups, cyclic groups, direct products, Fundamental theorem on finite abelian groups, Lagrange's theorem, normal subgroups and quotient groups, homomorphism, isomorphism and automorphism, Permutation Groups, alternating groups, Cayley's Theorem, Conjugate class, class equation, Sylow's Theorem, solvable groups

Rings, Integral domain, Field, Homomorphism, Kernal, isomorphism, ideals and quotient rings, principal ideal ring. Euclidean Ring, prime and irreducible elements, unique factorization domain, Polynomial Rings, Division Algorithm, irreducible polynomial, primitive polynomial

Field Extensions, Cyclotomic Polynomials, Galois theory and Insolvability of polynomials

4 Linear Algebra and Functional Analysis

Vector spaces: Definition and example, linear dependence and independence, Basis, dimension, subspaces, homomorphism, isomorphism

Linear Transformations: The algebra of Linear Transformation, Rank and Nullity, singular and non singular transformations, characteristic polynomials, minimal polynomials, Eigen values and eigen vectors, Annihilators, Matrix of Linear Transformation Examples, matrix of change of basis, similar matrices and Canonical Forms, Similar transformations, Invariant subspaces, Diagonalisation and Triangulation

Fundamentals of Normed Spaces, Bounded Linear Maps on Banach spaces, Compact Operators

Inner Product Spaces: Schwarz inequality, orthonormal basis, Gram-Schmidt orthogonalization process, orthogonal complement, Bounded Operators on Hilbert Spaces

5 Mathematical Analysis

Real Number System: Ordered sets, Fields, Real field, Extended real number system, Euclidean spaces

Basic Topology: Metric spaces, Open sets, Closed sets, Compact sets, Perfect sets, Connected sets

Numerical Sequence and Series: Convergent sequences, subsequences, Cauchy sequences, some special sequences. Series, Series of non-negative series, summation by parts, absolute convergence, addition and multiplication of series

Continuity: Limits of function, Continuous function, Continuity and Compactness, Continuity and Connectedness, Discontinuity, Monotonic functions

Differentiation: The derivative of a real function, Mean value theorems, the continuity of derivatives, Derivatives of higher order, Taylor's theorem, Differentiation of vector valued functions

The Riemann–Steiltje's Integral: Definition and existence of the integral, Properties of the integral, Integration and Differentiation

Sequences and Series of Functions: Pointwise and uniform convergence, Uniform convergence and continuity, Uniform convergence and integration, Uniform convergence and differentiation

Functions of Several Variables: Linear transformations, invertible linear operators, matrix representation, Differentiation, partial derivatives, gradients, directional derivative, continuously differentiable functions, The contraction principle, The Inverse and Implicit Function Theorems, Jacobians, Derivatives of Higher order and differentiation of integrals

6 Measure Theory and Integration

Lebesgue Measure, measurable sets and Lebesgue measure, algebra of measurable sets, countable subadditivity, countable additivity and continuity of measure, Borel sets, a non- measurable set

Measurable Function, Characteristic function, Sums, products and compositions, Sequential point wise limits, Simple functions, Littlewood's three principles

Lebesgue Integral of Bounded Functions: The Riemann integral, integral of simple functions, integral of bounded functions over a set of finite measure, bounded convergence theorem

The General Lebesgue Integral: Lebesgue integral of measurable nonnegative functions, Monotone convergence theorem, the general Lebesgue integral, integrable functions, linearity and monotonicity of integration, additivity over the domains of integration. Lebesgue dominated convergence theorem

Differentiation and Integration: Differentiation of monotone functions, Vitali covering lemma, Dini derivatives, Lebesgue differentiation theorem, functions of bounded variation

7 Differential Equations

First Order Linear Differential Equations, separable equations, exact equations, Bernoulli's equation and method of substitutions

Higher Order Linear Differential Equations: Homogeneous equations and general solutions, Initial value problems, existence and uniqueness of solutions, linear dependence and independence of solutions, Solutions of nonhomogeneous equations by Method of Variation of parameters, Method of Undetermined Coefficients, Linear equations with variable coefficients

Oscillations of Second Order Equations: Oscillatory and non Oscillatory differential equations, Boundary value problems, Sturm Liouville theory; Green's function. Solution in Terms of Power Series: - Solution near an ordinary point and a regular singular point–Frobenius method Legendre, Bessel's and Hypergeometric equations and their polynomial solutions, Rodregue's relation, generating functions, orthogonal properties, and recurrence relations

Successive Approximations Theory: Lipschitz condition, Convergence of successive approximations, Existence and Uniqueness theorem (Picard's theorem)

First Order Partial Differential Equations: Construction and Solutions of First Order Partial Differential Equations, Solutions Using Charpit's Method, Method of Cauchy Characteristics, Method of Separation of Variables

Second Order Partial Differential Equations: Equations with Variable Coefficients, Canonical Forms

Parabolic, Hyperbolic and Elliptic Equations, Solutions by various Methods Duhamel's Principle, Solutions to Higher Dimensional Equations, Solutions in cylindrical and spherical coordinate systems

8 Complex Analysis

Complex Number System, Extended plane

Analytic Functions, Power Series, Conformal Mappings

Complex Integration, Cauchy's theorems, Cauchy's Integral Formula, Index of a point with respect to a closed curve

Local Properties of Analytic Functions, singularities, Taylors's theorem, zeros and poles, Laurent's Series

Maximum Modules Principle, Schwarz lemma and applications, Hadamard's three circles theorem

The Residue theorem, The argument principle, Rouche's theorem. Evaluation of the integrals, Cauchy principal value

9 Topology

Topological Spaces, Bases, Sub-bases, Continuity, Convergence, Uniform convergence theorem, Homeomorphism

Separation and Countability Axioms, Connectedness, Path connectedness, locally connectedness

Compactness, Converging properties, Lindelof spaces, Countable compactness; Sequential compactness, limit point compactness; Bolzano-Weierstrass property, Tychnoffs theorem

Urysohn's Theorem, Urysohn's Lemma, Tietze's extension theorem, Urysohn's metrization theorem

References

1. Research Methodology: Methods and Techniques : C. R. Kothari
2. Research Methodology : D K Bhattacharyya
3. Journey Through Genius- The Great Theorems of Mathematics: William Dunham
4. A First Course in Abstract Algebra (5th or 7th Edition) : John B. Fraleigh
5. Topics in Algebra: : I.N.Herstein
6. Introduction to Linear Algebra Gilbert Strang
7. Linear Algebra : Hoffman and Kunze
8. Principles of Mathematical Analysis : W. Rudin
9. Mathematical Analysis : T.M. Apostal
10. Real Analysis: H.L.Royden
11. Complex Analysis: L.V. Ahlfors
12. Functions of one Complex Variable : John B Conway
13. An Introduction to Ordinary Differential Equation : Eurl A. Coddington
14. Differential Equations With Applications and Historical Notes : George Simmons
15. Elements of Partial Differential Equations : I.N. Sneddon
16. A First Course in Topology: J.R. Munkres
17. Set theory and Logic: R Stoll; Dover publications
18. Functional Analysis: Balmohan V Limaye
19. Graph Theory: J. A Bondy and U S R Murty
20. Introduction to Analytic Number Theory: Apostal

Question Paper Pattern

PART-A

25 Multiple Choice Questions (with only one correct answer) of **2** marks each. **All** questions are to be answered.

PART-B

16 Short Answer Questions of **5** marks each. **10** questions are to be answered.

Exam duration : **2 hours**; Maximum Marks : **100**